Strain

When a body is subjected to stresses, the resulting deformations are called *strains*, and here we want to relate strain to the stress which causes it. But first we must set up the coordinate system for strain, to describe precisely how a piece of material is deformed, or strained. Strain is defined as the relative change (ie, the fractional change) in the shape of the body. First, consider a stress which acts in the $x$ direction only on a 1D elastic string (Figure 2).

![Figure 2. An elastic string is stretched: Point O is fixed, L moves to L', M moves to M'.](image)

The point L on the string moves a distance $u$ to the point L' after stretching, and point M moves a distance $u + \delta u$ to the point M'. The strain in the $x$ direction, termed $e_{xx}$, is then given by

$$
e_{xx} = \frac{\text{change in length of } LM}{\text{original length of } LM} = \frac{L'M' - LM}{LM} = \frac{\delta x + \delta u - \delta x}{\delta x} = \frac{\delta u}{\delta x}$$

In the limit when $\delta x \to 0$, the strain at L is

$$e_{xx} = \frac{\partial u}{\partial x}$$

![Figure 3. Deformation of a rectangle: Point L moves to L', M moves to M', and N moves to N'.](image)
To extend the analysis to two dimensions \( x \) and \( y \), we must consider the deformation undergone by a rectangle in the \( x-y \) plane (Figure 3).

Points L, M, N move to \( L', M', N' \) with coordinates

\[
\begin{align*}
L &= (x, y) & L' &= (x + u, y + v) \\
M &= (x + \delta x, y) & M' &= (x + \delta x + u \frac{\partial u}{\partial x}, y + \delta y + v \frac{\partial v}{\partial x}) \\
N &= (x, y + \delta y) & N' &= (x + u \frac{\partial u}{\partial y}, y + \delta y + v \frac{\partial v}{\partial y})
\end{align*}
\]

The strain in the \( x \) direction \( e_{xx} \) is then given by

\[
e_{xx} = \frac{\text{change in length of } LM}{\text{original length of } LM} = \frac{\delta x + \frac{\partial u}{\partial x} \delta x - \delta x}{\delta x} = \frac{\partial u}{\partial x}
\]

Likewise, the strain in the \( y \) direction is

\[
e_{yy} = \frac{\text{change in length of } LN}{\text{original length of } LN} = \frac{\partial v}{\partial y}
\]

These strains are called the **normal strains**, the fractional changes in length along the \( x \) and \( y \) axes. For three dimensions, the third normal strain is

\[
e_{zz} = \frac{\partial w}{\partial z}
\]

**Shear Strain**

In contrast to the normal strains, the **shear components of strain** measure the change in shape undergone by the rectangle, and can be thought of as a rotation. Imagine a line perpendicular to another along which pure normal strain occurs; this line will rotate by an amount (in radians) equal to

\[
e_{xy} = e_{yx} = \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)
\]

\( e_{xy} \), \( e_{yx} \), \( e_{yz} \), and their symmetric counterparts measure the internal angular distortion of each coordinate axis with respect to the other two directions.

**Volumetric Strain**

For a volume, the fractional increase in volume caused by deformation is called cubical dilation, and is written \( \Delta \). The volume of the original rectangular parallelepiped is \( V \), where
The volume of the deformed parallelepiped is $V + \delta V$, and is approximately

$$V + \delta V = (1 + e_{xx})(1 + e_{yy})(1 + e_{zz})\delta x \delta y \delta z$$

The cubical dilation $\Delta$ is then given by

$$\Delta = \frac{\text{change in Volume}}{\text{original Volume}} = \frac{V + \delta V - V}{V} = \frac{(1 + e_{xx})(1 + e_{yy})(1 + e_{zz})\delta x \delta y \delta z - \delta x \delta y \delta z}{\delta x \delta y \delta z}$$

Since here we are considering infinitesimal strain, any product of strains can be dropped. So, cubical dilation is given by

$$\Delta = e_{xx} + e_{yy} + e_{zz}$$

$$\Delta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

$$\Delta = \nabla \cdot U$$

**Relationship between Stress and Strain**

$$\sigma_{xx} = ce_{xx}$$

In practice, in a given situation, we want to calculate the strains when we know already what the stresses are. In 1676, the English physicist Robert Hooke proposed that, for small strains, any strain is proportional to the stress that produces it. This is known as Hooke’s law and forms the basis of the theory of perfect elasticity. In one dimension, Hooke’s Law means that

where $c$ is a constant. Extending this to 3 dimensions gives us 36 different constants:

$$\sigma_{xx} = c_1 e_{xx} + c_2 e_{xy} + c_3 e_{xz} + c_4 e_{yy} + c_5 e_{yz} + c_6 e_{zz}$$

$$\vdots$$

$$\sigma_{zz} = c_{31} e_{xx} + c_{32} e_{xy} + c_{33} e_{xz} + c_{34} e_{yy} + c_{35} e_{yz} + c_{36} e_{zz}$$

If we assume that we are considering only isotropic materials (materials with no lateral variation), then the number of constants reduces to TWO:

$$\sigma_{xx} = (\lambda + 2\mu)e_{xx} + \lambda e_{yy} + \lambda e_{zz} = \lambda \Delta + 2\mu e_{xx}$$

$$\sigma_{yy} = \lambda \Delta + 2\mu e_{yy}$$

$$\sigma_{zz} = \lambda \Delta + 2\mu e_{zz}$$
The constants $\lambda$ and $\mu$ are known as the two Lamé constants (named after the nineteenth century French mathematician, G. Lamé). In suffix notation, these equations are written

$$\sigma_{ij} = \lambda \delta_{ij} + 2\mu e_{ij}$$

and where the Kronecker delta

$$\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$$

The Lamé constant $\mu$ is a measure of the resistance of a body to shearing strain and is often termed the shear modulus or the rigidity modulus. The shear modulus of a liquid or gas is zero. $\lambda$ does not have any easily intuitive physical interpretation.

Beside the Lamé elastic constants, other elastic constants are also used:

Young's Modulus $E$ is the ratio of tensional stress to the resultant longitudinal strain for a small cylinder under tension at both ends. Let the tensional stress act in the $x$ direction on the end face of the small cylinder, and let all the other stresses be zero. Then

$$\sigma_{xx} = \lambda \Delta + 2\mu e_{xx}$$

$$0 = \lambda \Delta + 2\mu e_{yy}$$

$$0 = \lambda \Delta + 2\mu e_{zz}$$

and $0 = \sigma_{yz} = \sigma_{zx} = \sigma_{yz}$

Adding these equations gives

$$\sigma_{xx} = 3\lambda \Delta + 2\mu \Delta$$

and substituting the above gives

$$e_{xx} = (\lambda + \mu) \frac{\Delta}{\mu}$$

Hence, Young's Modulus is

$$E = \frac{\sigma_{xx}}{e_{xx}} = \frac{(3\lambda + 2\mu)\Delta\mu}{(\lambda + \mu)\Delta} = \frac{(3\lambda + 2\mu)\mu}{(\lambda + \mu)}$$

Poisson's Ratio $\sigma$ (also named after another 19th century French mathematician, Simeon Denis Poisson) is defined as the negative of the ratio of the fractional lateral contraction to the fractional longitudinal extension for the same small cylinder under tension at both ends. Using the above equations, Poisson's ratio is given by

$$\sigma = -\frac{e_{yy}}{e_{xx}} = \frac{\lambda \Delta}{2\mu (\lambda + \mu)\Delta} = \frac{\lambda}{2(\lambda + \mu)}$$
The Bulk Modulus or incompressibility $K$ is the ratio of pressure to compression. Consider a small body subjected to a hydrostatic pressure (i.e., the body is immersed in a liquid). This pressure causes compression of the body. The ratio of the compressive pressure to the resulting compression is called the bulk modulus, and for a hydrostatic pressure $p$, the stresses are

$$
\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -p
$$

$$
\sigma_{xy} = \sigma_{yz} = \sigma_{xz} = 0
$$

Combining the above then gives:

$$
-p = \lambda \Delta + 2 \mu e_{xx}
$$

$$
-p = \lambda \Delta + 2 \mu e_{yy}
$$

$$
-p = \lambda \Delta + 2 \mu e_{zz}
$$

and

$$
e_{xy} = e_{yz} = e_{xz} = 0
$$

Adding these together gives

$$-3p = 3\lambda \Delta + 2\mu \Delta$$

Finally, the bulk modulus is given by

$$K = \frac{\text{Pressure}}{\text{Compression}} = \frac{\text{Pressure}}{\text{dilatation}}$$

$$= \frac{p}{-\Delta}$$

$$= \lambda + 2/3 \mu$$

Using these relations between the five elastic constants, we can write the stress/strain relation equations above in terms of any pair of these 5 constants.

Poisson’s ratio is dimensionless, positive, and less than 0.5 (0.5 for a liquid since $\mu = 0$), Young’s modulus, the Lamé constants, and the bulk modulus are all positive and are all generally quoted in units of Pascals. The two Lamé constants have almost the same value for rocks, so the approximation that $\lambda = \mu$ is sometimes made. This approximation is called Poisson’s relation.