Elasticity Theory

A large part of geophysics concerns understanding how material deforms when it is squeezed, stretched, or sheared. Elasticity theory is the mathematical framework which describes such deformation. By elastic, we mean that the material rebounds to its original shape after the forces on it are removed; a rubber eraser is a good example of an elastic material. Play-dough, on the other hand, is a good example of a material which is not elastic- if you squeeze it and then let go, the material stays in the shape it was when you squeezed it. Materials of this type are described as being plastic, rather than elastic, and elasticity theory is not suitable for understanding how these plastic materials deform. In reality, most materials are both elastic and plastic, but on different time scales. Silly-putty is one such example- it stretches like hot bubble gum, which is plastic behavior, but you can also roll it up and bounce it like a ball, which means it’s also elastic. The key is the time duration of the external force- when you stretch it like gum, the force is acting over several seconds, while when you bounce it off the floor, the impact on the floor lasts only a fraction of a second. So silly putty is plastic for forces acting over time scales of several seconds, by is elastic for very short –duration forces. Most of the materials that make up the earth are in fact elastic over time scales of years, and plastic over time scales of thousands to millions of years.

Stress

Before we describe how a medium deforms under an external force, we need to be able to describe precisely these external forces in terms of how they are oriented. Stress is defined as the force per unit area. When a deforming force is applied to a body, the stress is the ratio of the force to the area over which it is applied. If a force of 1 Newton(N) is applied uniformly to an area of 1 square meter, the stress is $1 \text{ N}/\text{M}^2 = 1 \text{ (Kg x M/}S^2)/\text{M}^2 = 1 \text{ Kg/(M S}^2) = 1 \text{ Pascal.}$ Your intuition should tell you that it is stress, rather than just force, which determines how a material deforms- think of an elephant inside a fancy ballroom, in one case standing on its feet, and in another case balanced on the tip of a needle. Clearly, the needle is going to puncture the floor much more effectively than the elephant’s feet. The reason is because although the force is the same, ie , the mass of the elephant times gravity, the area over which it is distributed is much smaller in the case of the needle, which, since stress is force/area, means the stress is by far greater in the case of the needle.
There are two basic types of stress. If the force is perpendicular (normal) to the surface over which it is acting, then the stress is termed a normal stress; if it is tangential to the surface, it is called a shear stress. Usually, the force is neither entirely normal nor tangential, but is at some arbitrary intermediate angle, in which case it can be resolved into components which are both normal and tangential to the surface; so the stress is composed of both normal and shearing components. The sign convention is that tensional stresses are positive and compressional stresses negative (in geophysics).

**Directionality**

Stress also has directionality to it- clearly a force acting in one direction will deform an object differently than a force which is oriented perpendicular to it. So, we need to develop how we incorporate directionality into stress. To do this, we use a hypothetical cube of material, it can be a free standing cube, or one imbedded deep in a larger piece of material. Consider a small parallelipiped with sides $\delta x, \delta y, \delta z$, as shown in Figure 1 below, and imagine that it is being stressed by some external force. On each face,

![Figure 1](image)

Figure 1 The parallelipiped is in static equilibrium. Stresses acting on the shaded back face are (-$\sigma_{xx}$, -$\sigma_{xy}$, and -$\sigma_{xz}$). Stress acting on the front face are ($\sigma_{xx}$, $\sigma_{xy}$, and $\sigma_{xz}$).

the stresses can be resolved into components in the $x,y$, and $z$ directions. The stresses acting on the shaded face are -$\sigma_{xx}$, -$\sigma_{xy}$, and -$\sigma_{xz}$. The notation is that $\sigma_{xy}$ refers to the stress $\sigma$ acting in the $y$ direction on the face which is perpendicular to the $x$ axis. The normal stress is thus -$\sigma_{xx}$ and the shearing stresses are -$\sigma_{xy}$, and -$\sigma_{xz}$.

If the parallelipiped is to be in static equilibrium (not moving), then the stresses on opposite faces must balance, and there must be no net couple which would rotate the parallelipiped. This requires that the stresses on opposite faces be equal in magnitude and opposite in direction. The shearing stresses on opposite faces of the parallelipiped (eg, -$\sigma_{xy}$ and $\sigma_{xy}$ on the back and front faces as shown in Figure 1) provide a couple of forces, or couple, which will rotate the parallelipiped. Since the parallelipiped must not rotate, this couple must be balanced by the couple provided by the shearing stresses -$\sigma_{yx}$, and $\sigma_{yx}$ acting on the two side faces. This means that $\sigma_{xy}$ must equal $\sigma_{yx}$. The same conditions apply to the other shearing stresses: $\sigma_{xy} = \sigma_{yx}$, $\sigma_{xz} = \sigma_{zx}$, $\sigma_{yz} = \sigma_{zy}$ (ie, the stress tensor must be symmetric).
Scalars versus Vectors versus Tensors

Stress and Strain are both tensor quantities, which means that they are more than just simple scalars or vectors. Everyday scalar numbers are just that—numbers, the dollars in my wallet, the pounds that I weigh, etc. However, there are other types of quantities that require more information than just the amount—for example, the velocity of my Subaru headed eastbound on I-90. While I might be driving 65 mph, that doesn’t actually describe my motion—we need the direction also, since 60 mph east is very different from 60 mph west if I’m analyzing rear-ending a semi going eastbound 70 mph. So velocity is a vector, meaning it has both a value and a direction associated with it. Tensors are just an extension of this, it’s a number which has any number of directions associated with it. In the case of stress, it is a 2-direction tensor, containing the direction of the force and the direction of the perpendicular to the face on which the force is acting. Strain is likewise a tensor with two directions, or a rank 2 tensor. A physical quantity which had three directions associated with it would simply be a rank-3 tensor.